Questions about binary expansions

1. A number x has a terminating binary expansion when??

<u>Answer:</u> When $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$
$q = 2^k$ for some $k \in \mathbb{Z}$.

(Similar to decimals
$$2.3756 = \frac{23756}{10000} = \frac{23756}{10^4}.)$$

2. A number has a nonterminating, repeating binary expansion when??

<u>Answer:</u> When $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$
q has other prime factors besides 2.

<u>Similar in base 10:</u> When $x = \frac{p}{q}$,
q has other prime factors besides 2 & 5.

3. A number has a nonrepeating, ∞ binary expansion <u>when</u>?

<u>Ans</u> when x is irrational.

example from decimals

$$x = 2.3\overline{427}$$

$$10000x = 23427.\overline{427}$$

subtract $\quad 10x = \quad 23.\overline{427}$

$$9990x = 23404$$

$$\Rightarrow x = \frac{23404}{9990}$$

This idea shows any eventually repeating decimal is rational

→ Same idea would work for binary.

Going the other way, every repeating decimal written as a fraction is something like

$$\frac{\#\#\#\#}{999990000} = \frac{p}{(10^k-1)10^m}$$

$\underline{\text{every}}$ $\frac{a}{b}$, with a/b in lowest terms and b having factors besides 2 & 5, can be written like this.

eg $\quad \frac{15}{37} = \frac{15k}{37k} = \frac{p}{9990000}$

This is a result of Euler's Theorem from number theory.

some integer

$$\frac{a}{b} \rightarrow \qquad bk = (\underset{10}{B}^{n} - 1)\underset{10}{B}^{s}$$

Why can we do this?

if b & B are relatively prime (no common factors),

then $$B^{\phi(b)} - 1 = kb$$

$\underline{\phi(b)}$ is called the Euler totient fcn.

integer.

If b is prime $\phi(b) = b-1$.

$$\frac{1}{7} = 0.\overline{142857} \qquad \phi(7) = 6$$

6-terms repeating

$$\frac{1}{11} = 0.\overline{09} \qquad \phi(11) = 10$$

(# of repeating digits) = factor of $\phi(b)$.